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Degree of heterogeneity of thermal field—a method of evaluation

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Abstract—In this paper the authors tried to use the probabilistic method to investigate a thermal field with so-called homogeneity index H. It enables global evaluation of a degree of heterogeneity of thermal field. The paper presents a structure of H and exemplary calculation results for a model simulating unsteady thermal field in a perpendicular connection of two thick-walled plates, infinitely extensive.

INTRODUCTION

THERE are certain problems occurring in different branches of science but having the same mathematical basis. Measures of an arrangement of a finite set of n objects in a limited area S are the branch of mathematics which is connected, among others, with extragalactic astronomy, geography (problems of colonization), economics and fluid mechanics [1–6].

Homogeneity belongs to a group of many features of arrangement of a finite set of objects. In ref. [7] a so-called factor of four arrangement features (concentration, clustering, weak anisotropy and strong anisotropy) has been constructed. In the 1950s, the method of Zwicky [8] was applied in astronomy for evaluation of tendency to grouping of galaxies and investigations of size of their clusters. In ref. [9] one can find a comparison of some different factors of distribution of galaxy populations, taken from *Catalogue of Jagiellonian Field* [10]. In refs. [5, 6] the original method by Garncarek was used for the description of the structure of two-phase flows.

In this paper the authors attempt to use the probabilistic method to investigate a thermal field with the so-called homogeneity coefficient H. It enables global evaluation of a degree of heterogeneity of thermal fields.

For the determination of thermal fields in solids with complicated shapes the finite-difference method (especially the method of Waniczew [11]) is widely applied. It was used in refs. [12, 13] for the determination of the transient field in the area of connection of a connector pipe with a chemical apparatus (connection of two thick-walled cylinders with perpendicular axes). The calculated thermal field was the basic point for determination of the quasi-steady field of thermal stresses in this connection. The moment of heating of the connection, when the maximum value of temperature gradient occurred, was assumed to be the criterion.

The temperature gradient is a local value in the thermal field, so the criterion of maximum gradient does not mean the maximum turbulence in this field. The detailed analysis of all local values of the gradient in thermal field enables evaluation of a degree of its heterogeneity.

CONSTRUCTION OF HOMOGENEITY INDEX OF THERMAL FIELD, *H*

Homogeneity of thermal fields belong to their most important features. In this paper the original method of measurement of homogeneity of thermal fields is proposed—based on probabilistics considerations.

The thermal field which can be illustrated by a map having area S (Fig. 1) is homogeneous to scale S/κ if:

$$\Lambda_i T_i = \frac{N}{\kappa} \tag{1}$$

where

$$N = \sum_{i=1}^{\kappa} T_i. \tag{2}$$

Thus the homogeneity of the thermal field depends on a range of considerations. In practice non-homogeneous thermal fields usually occur. There is a very important problem: what field is more or less non-homogeneous? For thermal fields for which

$$\sum_{i=1}^{\kappa} T_i = \text{const.} \tag{3}$$

a measure of homogeneity can be expressed as

NOMENCLATURE

- a coefficient of temperature equalization [$m^2 s^{-1}$]
- c_{pi} specific heat, at constant pressure, of difference element i [$J kg^{-1} K^{-1}$]
- $E(X)$ expected value of random variable X [—]
- $E(\mu)$ expected value of random variable μ [—]
- F field of events [—]
- GT temperature gradient [$K m^{-1}$]
- H homogeneity index [—]
- N sum of temperatures in the investigated area S [K]
- $P^N_{T_1,\ldots,T_\kappa}$ probability of obtaining a map of thermal field with size κ [—]
- p_i probability of event for random variable X, taking values $x_i \in R$ [—]
- $\dot{Q}_{ji}$ heat flux between difference element i and adjacent element j [W]
- $\dot{Q}_{Fi}$ heat flux between fluid and difference element i [W]
- R set of real numbers [—]
- R_{ji} resistance of heat conductivity between difference element i and adjacent element j [$m K W^{-1}$]
- R_{Fi} resistance of heat penetration between fluid and difference i [$m K W^{-1}$]
- T_i temperature of the ith subdomain (difference element) on the map of thermal field [K]
- T_{Fi} temperature of fluid washing difference element i [K]
- V_i volume of difference element i [m^3]
- X random variable [—].

Greek symbols

- α minimum value of temperature in the considered area S [K]
- δ plate thickness [m]
- κ number of subdomains (difference elements) with the identical fields of area S [—]
- μ random variable (homogeneity measure)
- ρ_i density of difference element i [$kg m^{-3}$]
- τ time [s]
- $\Delta\tau$ time step [s]
- ω elements of space of elementary events [—]
- Ω space of elementary events [—].

$$\mu = \sum_{i=1}^{\kappa} \left(T_i - \frac{N}{\kappa} \right)^2. \tag{4}$$

For thermal fields presented in Fig. 2 we obtain $\mu = 12$, $\mu = 5$, $\mu = 4$, $\mu = 2$, $\mu = 0$ respectively. The measure μ takes a value equal to zero for the homogeneous field and increases with the heterogeneity of the field.

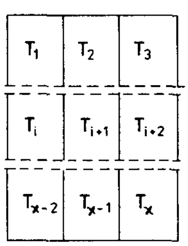

FIG. 1. The thermal field.

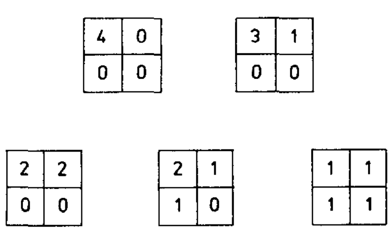

FIG. 2. Example of thermal field for $N = 4$ K.

For thermal fields presented in Fig. 3, which do not satisfy the condition expressed by (3), we obtain $\mu = 12$ and $\mu = 14$, respectively. From the obtained results it appears that the second of the considered fields is more heterogeneous than the first one but it is difficult to accept this result. From intuition it appears that the result should be the opposite. Thus, μ cannot be used for evaluation of homogeneity of fields which do not satisfy the condition expressed by (3). In practice, the compared fields usually do not satisfy the condition (3).

In this paper a measure of homogeneity H, enabling comparison of thermal fields which do not strictly satisfy the condition (3), is proposed. It is expressed by

$$H = \frac{\mu}{E(\mu)} \tag{5}$$

where $E(\mu)$ is the expected value of random variable μ. It should be stated that both μ and its expected value $E(\mu)$ are expressed in the same unit, so the measure of homogeneity H is a number without designation. The structure of the measure is based on pro-

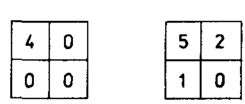

FIG. 3. Example of thermal field for difference N.

babilistic considerations, described below without any proof.

Evaluation of homogeneity of different thermal fields, shown by maps (see Fig. 1), is the aim of the considerations. It is assumed that:

1. A number κ of subdomains with identical areas, forming the investigated area S, is the same for each map.

2. $T_1, \ldots, T_\kappa$ are natural numbers.

The postulate 2 is necessary for generating a probabilistic model. Then, while investigating real thermal fields, it is omitted. Let us consider one map of a thermal field. It can be understood as a result of an experiment consisting of placing N balls in κ chambers. If it is assumed that each ball can be found in the chamber with the same probability equal to $1/\kappa$, then the probability of obtaining T_i balls in the ith chamber for $i = 1, \ldots, \kappa$ and the probability of obtaining a map as in Fig. 1, can be expressed as

$$P^N_{T_1,\ldots,T_\kappa} = \frac{N!}{T_1!*\ldots*T_\kappa!}\frac{1}{\kappa^N}. \tag{6}$$

All maps, which can be obtained as a result of arrangement of a given number of balls, N, in a given number of chambers, κ, are a space of elementary events Ω. Elements of the space are denoted by $\omega = (T_1, \ldots, T_\kappa)$. A family of all subsets of the space Ω is a field of events F. (Ω, F, $P^N_{T_1,\ldots,T_\kappa}$) is a probabilistic space.

Measure μ, defined by equation (4), is a function determined on the space Ω with values in a set of real numbers R, so it is a random variable. Determination of the expected value $E(\mu)$ of the random variable μ is difficult. As for the finite space of elementary events $\Omega = (\omega_1, \ldots, \omega_m)$, the expected value $E(X)$ of random variable $X:\Omega \to R$ is expressed by

$$E(X) = \sum_{i=1}^{m} x_i p_i \tag{7}$$

where $p_i = P(X = x_i)$ is the probability of the event that random variable X takes values $x_i \in R$. Random variable $\mu:\Omega \to R$ assigns a real number to each elementary event $\omega = (T_1, \ldots, T_\kappa)$

$$\sum_{i=1}^{\kappa}\left(T_i - \frac{N}{\kappa}\right)^2 \tag{8}$$

with probability given by equation (6)

$$E(\mu) = \sum_{\substack{T_1,\ldots,T_\kappa \\ \sum_{i=1}^{\kappa} T_i = N}} \left[\sum_{i=1}^{\kappa}\left(T_i - \frac{N}{\kappa}\right)^2\right]\frac{N!}{T_1!*\ldots*T_\kappa!}\frac{1}{\kappa^N}. \tag{9}$$

Thus we obtain:

$$E(\mu) = \frac{N(\kappa-1)}{\kappa}. \tag{10}$$

Correctness of this equation can be easily verified with a computer for chosen but not very high N and κ.

Thus, homogeneity index H is expressed by

$$H = \frac{\kappa}{(\kappa-1)N}\sum_{i=1}^{\kappa}\left(T_i - \frac{N}{\kappa}\right)^2 \tag{11}$$

and, after simple transformations, we have

$$H = -\frac{N}{\kappa-1} + \frac{\kappa}{(\kappa-1)N}\sum_{i=1}^{\kappa} T_i^2. \tag{12}$$

Homogeneity index H, constructed in this paper, is equal to zero for thermal fields which are perfectly homogeneous (in each area of the network $T_i = N/\kappa$). When heterogeneity of thermal field increases, H increases and reaches the maximum value

$$H_{\max} = N. \tag{13}$$

For maps from Fig. 3 values $H = 4.0$ and $H = 2.3$ are assumed, respectively. It means that the thermal field illustrated by the first map (Fig. 3) is more heterogeneous than that illustrated by the second map (Fig. 3).

From the above example it appears that H can be used for comparison of different thermal fields, but they must fulfill the condition (3).

The results obtained agree with intuition (and a real state). However, it is not possible to compare two different thermal fields, which do not satisfy the condition H, because of variable range of H, dependent on N.

In practice, the considered thermal fields, often do not satisfy condition (3). In such a case we propose to make computations for reduced thermal field. It is obtained by subtraction of

$$\alpha = \min_{i\in(1,\ldots,\kappa)} T_i$$

from each value of T_i for $i = 1, 2, 3, \ldots, \kappa$.

The reduced thermal field corresponds to the gradient field, identical with the gradient field corresponding to a real thermal field, used for construction of the reduced thermal field.

DIFFERENCE SIMULATION OF UNSTEADY THERMAL FIELD

The equation of unsteady thermal conduction

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T \tag{14}$$

is a starting point for the determination of a thermal field in an isotropic body without any internal heat sources. It gives a general relation between temperature, time and the coordinates of space, which must be satisfied for each thermal field.

In this paper, the computational example of the unsteady thermal field concerns a complicated shape (perpendicular connection of two thick-walled plates

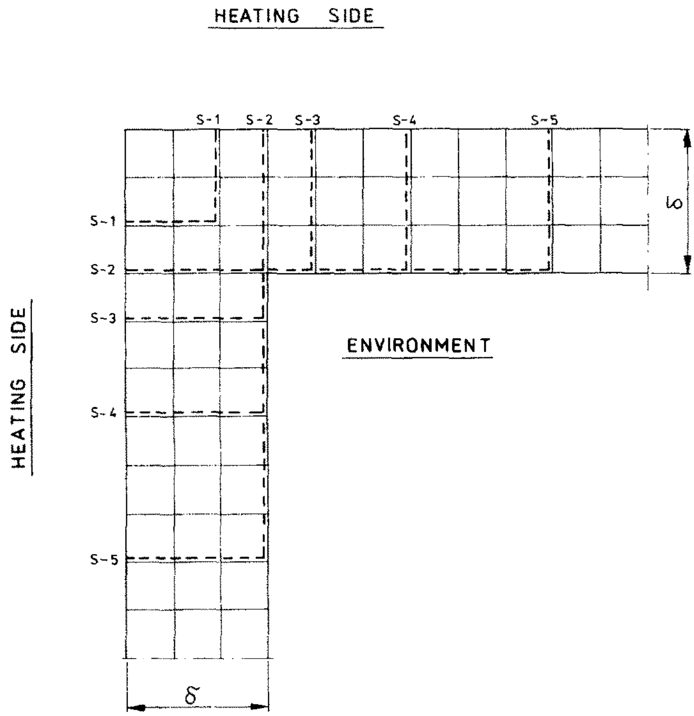

FIG. 4. A perpendicular connection of two thick-walled plates infinitely extensive.

(Fig. 4), infinitely extensive). Analysis of the unsteady thermal field in the connection was carried out taking into account occurrence of temperature gradients GT and heterogeneity characterized with H.

For obtaining an approximate description of complex two-dimensional processes, occurring in the connection and resulting from heating, it was necessary to take simplifying assumptions:

- connection between the plates is a homogeneous body without internal heat sources;
- ends of the plates are adiabatic surfaces, being at a sufficient distance from the connection; they are placed in the area where repeatability of thermal field is observed;
- physical parameters of the material for the plates do not depend on temperature;
- boundary condition of the third type occurs on the plate surface;
- parameters of fluids in the environment of the connection are constant.

The geometric form of the connection was complex, so while using the assumptions for difference approximation of equation (14), the method of elementary balances was applied [11].

A sum of heat streams flowing to the considered node of the difference element (volume V_i) from adjacent nodes j and from the body surface (Fig. 5) influences increase of enthalpy of the difference element

$$\sum_j \dot{Q}_{ji} + \dot{Q}_{Fi} = V_i \rho_i c_{pi} \frac{\Delta T_i}{\Delta \tau}. \tag{15}$$

A set of equations for the energy balance of type (15) for all different elements enables determination of the temperature of nodes versus time.

Using the difference quotient, from energy balance (15), equations for the calculation of temperature of the element (node) after a time step $\Delta\tau$ were obtained on the basis of temperature at time τ

$$T_{i,\tau+\Delta\tau} = T_{i,\tau}\left[1 - \frac{\Delta\tau}{V_i\rho_i c_{pi}}\left(\frac{1}{R_{Fi}} + \sum_j \frac{1}{R_{ji}}\right)\right] + \frac{\Delta\tau}{V_i\rho_i c_{pi}}\left(\frac{T_{Fi}}{R_{Fi}} + \sum_j \frac{T_{i,\tau}}{R_{ji}}\right). \tag{16}$$

In order to obtain physical correctness of (16), the length of the time interval $\Delta\tau$ was specially chosen (a term in the brackets should not be negative). Thus a time step is limited

$$\Delta\tau \leqslant \frac{V_i\rho_i c_{pi}}{\frac{1}{R_{Fi}} + \sum_j \frac{1}{R_{ji}}} = \Delta t_{\min}. \tag{17}$$

Using (17) it was possible to calculate the maximum time interval in each node, next $\Delta\tau$ not greater than the minimum boundary interval was assumed for calculations.

The influence of changes in fluid temperature on thermal field in the connection was characterized, after each time step $\Delta\tau$, by the maximum value of the temperature gradient and its mean arithmetic value. At the same time a value of homogeneity index was calculated for thermal field after each step $\Delta\tau$.

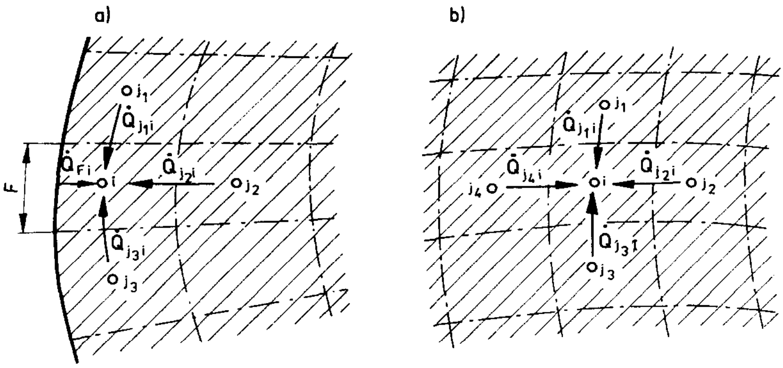

FIG. 5. Energy balance for differential element: (a) outer, (b) inner.

RESULTS OF CALCULATIONS

Calculations were made in order to obtain a thermal field under unsteady state, disturbed by a shape of the connection. For the obtained thermal field the behaviour of the maximum temperature gradient was observed in all the areas of calculations as well as its mean value in the limited ranges starting from the zone *S*-1 to *S*-5 (Fig. 4). Homogeneity index was observed in a similar way. The obtained results of calculations are shown in Figs. 6 and 7. Calculations were made for the following data:

- fluid temperature 573 K;
- the same plate thickness 0.090 m;
- thermal properties of material for the plates: coefficient of heat conductivity 31.2 W m^{-1} K^{-1}, specific heat capacity 461.0 J kg^{-1} K^{-1}, density 7780.0 kg m^{-3};
- coefficient of heat penetration at the heating side 100.0 at the side of heat giving up 15.0 W m^{-2} K^{-1};
- temperature at the side of heat giving up 0°C.

CONCLUSIONS

The proposed homogeneity index, applied for evaluation of the heterogeneity degree of unsteady thermal field, represents its disturbance very well. It reaches the maximum value (a degree of departure from homogeneity) for the maximum value of temperature gradient in all the computational areas, as well as for mean values of the gradient in computational zones from *S*-1 to *S*-5 (Figs. 6 and 7).

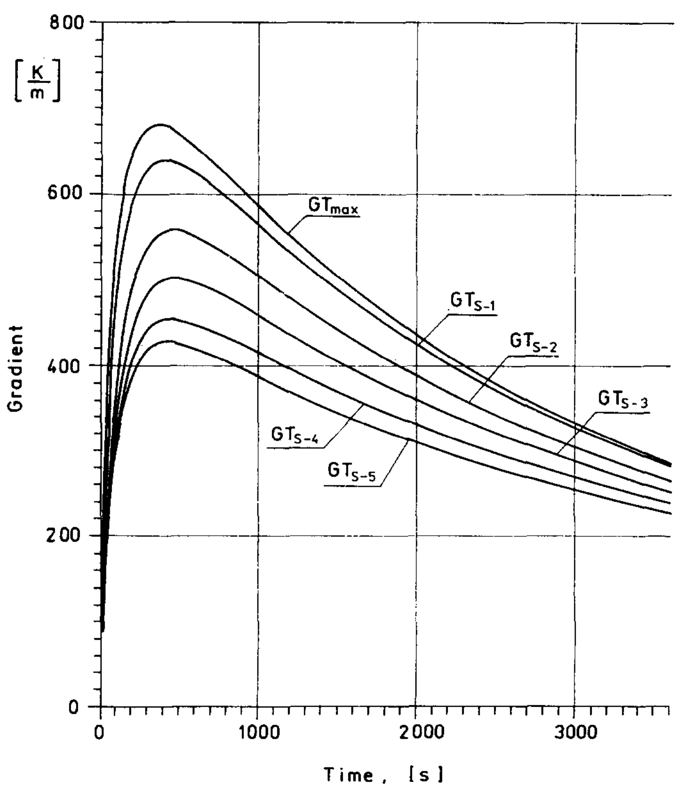

FIG. 6. Maximum and mean temperature gradient in thermal field.

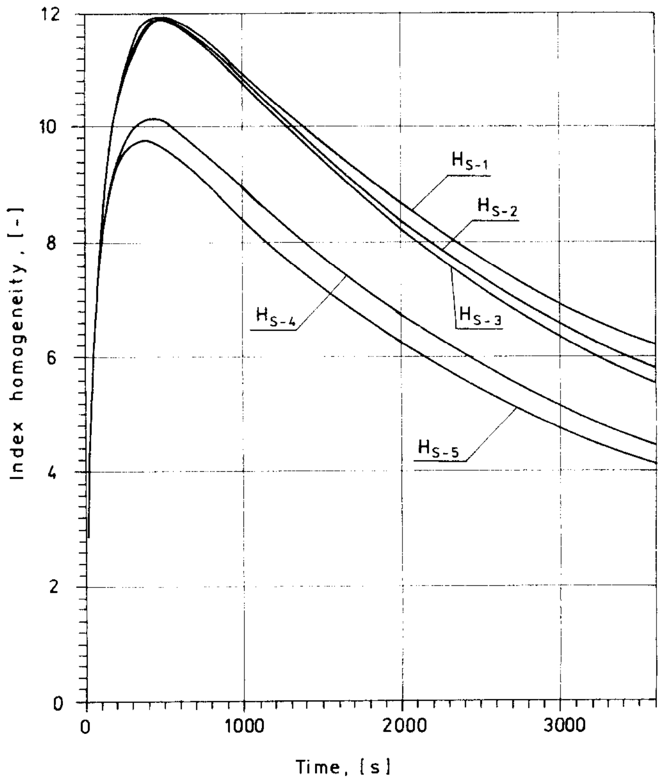

FIG. 7. Index homogeneity in thermal field.

Investigations of the thermal field, consisting of calculating some certain parameters for a series of subdomains $S\text{-}1 \subset S\text{-}2 \subset S\text{-}3 \subset S\text{-}4 \subset S\text{-}5$ of this field, enable localization of subdomains influencing the maxima of the investigated parameters. The index H identifies with the area in which the maximum value of temperature gradient occurred (Fig. 7—intersection of $H_{S\text{-}2}$ and $H_{S\text{-}3}$) better than the mean value of the gradient in those subdomains. From the exemplary results it appears that H can replace a local physical quantity, i.e. temperature gradient, in evaluation of disturbances of thermal fields.

The global approach to evaluation of thermal field with the factor H is a new one in thermodynamics. Owing to this, it is possible to evaluate departures from homogeneity of the considered thermal fields. It may be a device for predicting a way of heating or cooling real elements. With this one dimensionless number—H—it is possible to evaluate (in a precise way) heterogeneity of any thermal field.

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DEGRE D'HETEROGENEITE DES CHAMPS THERMIQUES—UNE METHODE D'EVALUATION

Résumé—Les auteurs utilisent la méthode probabiliste pour étudier le champ thermique avec ce qui est appelé l'index d'homogénéité H. Il permet l'évaluation globale d'un degré d'hétérogénéité du champ thermique. On montre une structure de H et un exemple de calcul pour un modèle qui simule le champ thermique variable dans une convection perpendiculaire de deux plaques épaisses infiniment étendues.

EIN VERFAHREN ZUR ABSCHÄTZUNG DES HETEROGENITÄTSGRADES THERMISCHER FELDER

Zusammenfassung—In der vorliegenden Arbeit wird ein Wahrscheinlichkeits-Verfahren auf die Untersuchung von Temperaturfeldern angewandt. Der sogenannte Homogenitätsindex H ermöglicht eine globale Abschätzung des Heterogenitätsgrades des Temperaturfeldes. Die Struktur von H wird vorgestellt und außerdem ein beispielhaftes Rechenergebnis für ein Modell des instationären Temperaturfeldes in einer rechtwinkligen Verbindung zwischen zwei dickwandigen, unendlich ausgedehnten Platten.

СТЕПЕНЬ НЕОДНОРОДНОСТИ ТЕПЛОВОГО ПОЛЯ-МЕТОД ОЦЕНКИ

Аннотация—В настоящей работе предпринята попытка использования вероятностного метода исследования теплового поля с помощью так называемого коэффициента неоднородности H. Метод позволяет оценивать степень неоднородности теплового поля. Приводятся структура Н и результаты расчетов на основе модели, описывающей нестационарное тепловое поле при перпендикулярном соединении двух пластин большой толщины.